Internship proposal: Causal decompositions of unitary processes

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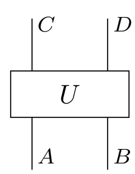
Background

In recent years, the field of quantum foundations has seen a surge in interest for the subject of *quantum causal modelling*: pinning down how the notion of causal influence could be conceptually understood and mathematically formalised in quantum theory, given the failure of classical notions of causality, as exemplified by the violation of Bell inequalities [1, 2, 3]. The hope is to better understand quantum theory's core structure, and therefore what it tells us about the world.

In that context, a particularly important conjecture has emerged, that of the existence of *causal decompositions* [4]. It concerns the relationship between two core structures of quantum theory:

1. *Causal structure*, on the one hand, is about empirically detectable influences from one quantum system to another in a unitary evolution. For instance, given a unitary $U : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_C \otimes \mathcal{H}_D$ from two systems A and B to two systems C and D, a possible causal structure for U would be that it features no causal influence from A to D, i.e. that operations performed on A before U takes place do not affect the outcome probabilities in measurements performed on D after U takes place. We can denote this as:

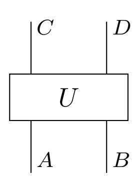


$$: \quad A \not\to D\,. \tag{1}$$

Note how causal structure involves *operational quantities*, i.e. the statistics that can be collected in concrete experiments. It is about what can be probed in a lab with access to U.

2. *Compositional structure*, on the other hand, is about the possibility for a certain unitary map to be seen as the mathematical composition of 'smaller' unitary maps. For instance, a possible compositional structure for the previous gate U would be the existence of the following decomposition:

$\exists U_1, U_2,$



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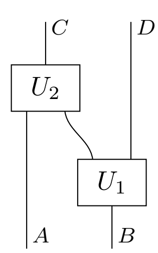

$. \qquad (2)$

Compositional structure is about *underlying mathematical structure*: it concerns handy ways in which we can write down the mathematical objects describing the dynamics of the world. It might also tell us about 'what is going on deep down', i.e. about the fine-grained dynamics underlying a given coarse-grained one. The downside of this is that compositional structure has no direct operational meaning, i.e. does not directly relate to what can be probed in a lab with access to U.

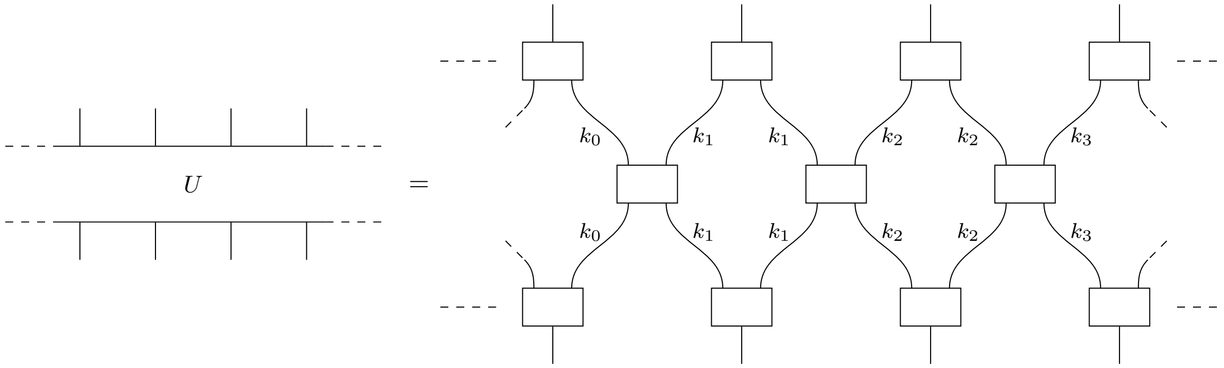

Figure 1: Causal decomposition of a 1D QCA with radius 1. A 1D QCA (as represented on the left) is a unitary on a 1D array of sites $\ldots, A_{-1}, A_0, A_1, \ldots$ such that a given input site only affects the output sites less than a certain causality radius away from it (for instance, in radius 1, A_0 only affects A_{-1}, A_0 and A_1, etc). The causal decomposition for a QCA with causality radius 1 is represented on the right (higher radii simply correspond to more layers). The k-superscripts correspond to the language of routed circuits.

(2) implies (1), since it directly shows that there is 'no path' from A to D. But remarkably, the reverse implication is also true, so (1) and (2) are equivalent statements [5]. The conjecture of causal decompositions is that this equivalence holds in general: for a unitary with arbitrarily many inputs and outputs, any given causal structure (i.e. a set of no-influence relations between the inputs and outputs) is equivalent to a corresponding compositional structure [4]. Proving this conjecture in general would be of great importance: it would unveil a remarkable correspondence between the operational and the mathematical structures of quantum theory, between 'what we can empirically probe' and 'what is going on deep down in the Universe'.

Investigating this conjecture has yielded important structural results about quantum theory already; for instance, it has shown the need to go beyond the formalism of unitary circuits to a more flexible one, that of unitary *routed* circuits. Recently, a major step forward has been attained with the proof that causal decompositions exist for *1D quantum cellular automata* (1D QCAs) – see Figure 1. This essentially shows that causally acceptable dynamics in a (discretised) 1+1D spacetime are all and only those consisted of a series of nearest-neighbours interactions. The proof techniques for this result involve the use of the theory of C*-algebras, with several new methods, and are of interest in themselves.[1]

The project

The previous result about causal decompositions of 1D QCAs has a caveat: it applies only to 1D QCAs over a finite array of sites – in other words, the series $\ldots, A_{-1}, A_0, A_1, \ldots$ cannot be infinite (it either stops at two ends, or is looped around). This limitation is due to the fact that the proof techniques have so far only been developed for *finite-dimensional* C* algebras, while an infinite array of sites would necessarily involve the consideration of infinite-dimensional ones. The shift to infinite dimension involves numerous mathematical complications, due to the appearance of convergence considerations.

The project would thus be for the student to investigate a generalisation of that result to infinite dimension. It would involve carefully analysing the finite-dimensional definitions and proofs and adapting them to the infinite-dimensional case. Although this will be quite technical, it is a somewhat safe project, in the sense that there is no reason to fear that the result wouldn't hold in infinite dimension. On the other hand, the outcome would still be of great significance and interest. Pinning down the generalisation of these new C*-algebraic techniques to infinite dimension would also be very likely to lead to application to other domains besides that particular outcome.

[1] The two papers describing the methods and the result will be on the arxiv soon; in the meantime, drafts are available upon request.

Profile of the student

The student should have followed a relevant curriculum in theoretical physics, mathematics, or computer science. This project requires a student with a solid mathematical background, and a willingness to dive into complicated maths. The physics is still there, and the project's outcome would hold direct physical significance, but the path might be somewhat dry. It is not imperative to have followed a course on operator algebras before, but that would of course be welcome.

The internhip could continue into a PhD.

Location

The internship will take place at Télécom Paris in the Quriosity team, supervised by Augustin Vanrietvelde. Quriosity brings together physicists and computer scientists to explore the fundamental informational aspects of quantum theory.

References

[1] R. W. Spekkens, "The paradigm of kinematics and dynamics must yield to causal structure," `arXiv:1209.0023 [quant-ph]`.

[2] J.-M. A. Allen, J. Barrett, D. C. Horsman, C. M. Lee, and R. W. Spekkens, "Quantum common causes and quantum causal models," *Physical Review X* **7** (2017) 031021, `arXiv:1609.09487 [quant-ph]`.

[3] J. Barrett, R. Lorenz, and O. Oreshkov, "Quantum causal models," `arXiv:1906.10726 [quant-ph]`.

[4] R. Lorenz and J. Barrett, "Causal and compositional structure of unitary transformations," *Quantum* **5** (2021) 511, `arXiv:2001.07774 [quant-ph]`.

[5] T. Eggeling, D. Schlingemann, and R. F. Werner, "Semicausal operations are semilocalizable," *Europhysics Letters (EPL)* **57** no. 6, (2002) 782–788, `arXiv:quant-ph/0104027`.